Lesson 15: Operations on Relations

1. Set operations and relations

Relations are sets. All set operations are applicable to relations

Examples:

Let A= {3, 5, 6, 7}
B ={4, 5, 9}

Consider two relations R and S from A to B:

$R = \{(x,y) | x \in A, y \in B, x < y\}$

If $(x,y) \in R$ we write xRy

R is a finite set and we can write down explicitly its elements:
R= {(3,4),(3,5),(3,9),(5,9), (6,9),(7,9)}

$S = \{(x,y) | x \in A, y \in B, |x-y| = 2\}$

If $(x,y) \in S$ we write xSy

S is a finite set and we can write down explicitly its elements:
S = {(3,5), (6,4), (7,5), (7,9)}

For R and S the universal set is A x B:
{(3,4), (3, 5), (3, 9),
(5, 4), (5, 5), (5, 9),
(6, 4), (6, 5), (6, 9),
(7, 4), (7, 5), (7, 9)}

a) intersection of R and S:

$R \cap S = \{(x,y) \mid xRy \wedge xSy\}$ $R \cap S = \{(3,5),(7,9)\}$

b) union of R and S:

$R \cup S = \{(x,y) \mid xRy \vee xSy\}$

$R \cup S$ = { (3,4),(3,5),(3,9),(5,9), (6,9),(7,9), (6,4), (7,5) }

c) complementation:

$\sim R = \{(x,y) \mid \sim(xRy)\}$

$\sim R = U - R$

The universal set for R is the Cartesian product A x B

A = { 3,5,6,7}

B = {4,5,9}

U = A x B = {(3,4), (3,5), (3,9), (5,4), (5,5), (5,9),
(6,4), (6,5), (6,9), (7,4), (7,5), (7,9)}

R = {(3,4), (3,5), (3,9), (5,9), (6,9), (7,9)}

U - R = {(5,4), (5,5), (6,4), (7,4), (7,5)}

Note that for any two sets A and B, $A - B = A \cap \sim B$

d) difference R - S, S - R:

$R - S = \{(x,y) \mid xRy \wedge \sim(xSy)\}$

R - S = {(3,4),(3,9),(5,9), (6,9)}

2. Inverse relation

Let $R: A \to B$ be a relation from A to B. The inverse relation $R^{-1}: B \to A$ is defined as in the following way:

$$R^{-1}: B \to A \; \{(y,x) \mid (x,y) \in R\}$$

Thus $xRy \equiv yR^{-1}x$

Examples:

a. Let A = {1,2,3}, B = {1,4,9}

Let $R: B \to A$ be the set {(1,1), (1,4), (2,2), (2,4), (3,3)}

$R^{-1}: B \to A$ is the relation {(1,1), (4,1), (2,2), (4,2), (3,3)}

b. Let A = {1,2,3}, $R: A \to A$ be the relation {(1,2), (1,3), (2,3)}

R^{-1} is the relation: {(2,1), (3,1), (3,2)}

c. other examples:

R is $>$, R^{-1} is $<$ $5 > 8$, $8 < 5$

R : parent(x,y), R^{-1} : child(y,x)

3. Composition of relations

Let X, Y and Z be three sets, R be a relation from X to Y, S be a relation from Y to Z.

A composition of R and S is a relation from X to Z :

$S \circ R = \{(x,z) \mid x \in X, z \in Z, \exists y \in Y$, such that xRy, and $ySz\}$

Note that the operation is right-associative, i.e. we first apply R and then S

Example 1:
Let X, Y, and Z be the sets:

X: {1,3,5}
Y: {2,4,8}
Z :{2,3,6}

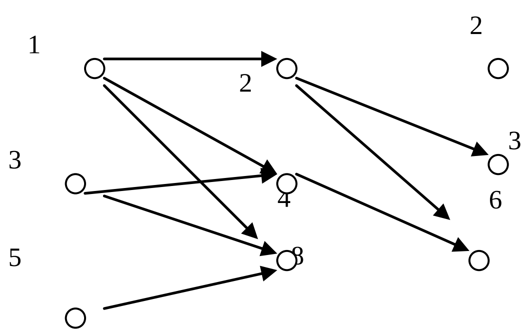
Let $R : X \rightarrow Y$, and $S : Y \rightarrow Z$, be the relation "less than":

R = {(1,2),(1,4),(1,8),(3,4),(3,8),(5,8)}
S = {(2,3),(2,6),(4,6)}

$S \circ R$:{(1,3), (1,6), (3,6)}

The element (1,3) is formed by combining (1,2) from R and (2,3) from S
The element (1,6) is formed by combining (1,2) from R and (2,6) from S
Note, that (1,6) can also be obtained by combining (1,4) from R and (4,6) from S.
The element (3,6) is formed by combining (3,4) from R and (4,6) from S

Graphical representation of the composition:



1 ($S \circ R$) 3 (intermediary 2)
1 ($S \circ R$) 6 (intermediary 2, also 4)
3 ($S \circ R$) 6 (intermediary 4)

Intermediary elements in Y - belong both to the range of R and the domain of S

Properties of composition: right-associative, not commutative.

4. Identity relation

Identity relation on a set A is defined in the following way:

$I = \{(x,x) | x \in A\}$

Example:

Let A = {a, b, c}, I = {(a,a), (b,b), (c,c)}

5. Problems:

Let A = {1, 2, 3}, B = {a, b}, C = {x, y, z}

a. Let R = {(1,a), (2,b), (3,a)} and S = {(a,y),(a,z),(b,x),(b,z)}

Find $S \circ R$

b. Let R = {(1,a), (2,b), (3,a)} and S = {(a,y),(a,z)}

Find $S \circ R$

a. Let R = {(1,a), (2,b)} and S = {(a,y), (b, y), (b,z)}
Find $S \circ R$

b. Let R = {(1,a), (2,b), (3,a)} and S = {(a,y),(a,z),(b,x),(b,z)}
Find R^{-1} , S^{-1} and $R^{-1} \circ S^{-1}$

Solutions

Let A = {1, 2, 3}, B = {a, b}, C = {x, y, z}

a. Let R = {(1,a), (2,b), (3,a)} and S = {(a,y),(a,z),(b,x),(b,z)}

Find $S \circ R$

Solution: {(1,y), (1, z), (2,x),(2,z), (3,y), (3, z)}

b. Let R = {(1,a), (2,b), (3,a)} and S = {(a,y),(a,z)}

Find $S \circ R$

Solution: {(1,y), (1, z), (3,y), (3, z)}

c. Let R = {(1,a), (2,b)} and S = {(a,y), (b, y), (b,z)}
Find $S \circ R$

Solution: {(1,y), (2,y), (2, z)}

d. Let R = {(1,a), (2,b), (3,a)} and S = {(a,y),(a,z),(b,x),(b,z)}
Find R^{-1} , S^{-1} and $R^{-1} \circ S^{-1}$

Solution:
R^{-1} = {(a,1), (b,2), (a,3)}
S^{-1} = {(y,a),(z,a),(x,b),(z,b)}

$R^{-1} \circ S^{-1}$ = {(y,1), (y,3), (x,2), (z,1), (z,3), (z,2)}