Lesson 11: SETS IDENTITIES

Using the operation unions, intersection and complement we can build expressions over sets.

Example:
A - set of all black objects
B - set of all cats

A \cap B - set of all black cats

The set identities are used to manipulate set expressions

\begin{align*}
A \cup \neg A &= U & \text{Complementation Law} \\
A \cap \neg A &= \emptyset & \text{Exclusion Law} \\
A \cap U &= A & \text{Identity Laws} \\
A \cup \emptyset &= A \\
A \cup U &= U & \text{Domination Laws} \\
A \cap \emptyset &= \emptyset \\
A \cup A &= A & \text{Idempotent Laws} \\
A \cap A &= A \\
\neg(\neg A) &= A & \text{Double Complementation Law} \\
A \cup B &= B \cup A & \text{Commutative Laws} \\
A \cap B &= B \cap A \\
(A \cup B) \cup C &= A \cup (B \cup C) & \text{Associative Laws} \\
(A \cap B) \cap C &= A \cap (B \cap C) \\
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) & \text{Distributive Laws} \\
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\
\neg(A \cap B) &= \neg A \cup \neg B & \text{De Morgan's Laws} \\
\neg(A \cup B) &= \neg A \cap \neg B \\
A - B &= A \cap \neg B & \text{Alternate representation for set difference}
\end{align*}
Proof problems for sets

A. Element Proofs

Definitions used in the proofs
Def 1: $A \cup B = \{ x | x \in A \lor x \in B \}$
Def 2: $A \cap B = \{ x | x \in A \land x \in B \}$
Def 3: $A - B = \{ x | x \in A \land x \not\in B \}$
Def 4: $\sim A = \{ x | x \not\in A \}$

Inference rules often used:

$P \land Q \models P, Q$
$P, Q \models P \land Q$
$P \models P \lor Q$

How to prove that two sets are equal:
A = B
1) show that $A \subseteq B$, i.e. choose an arbitrary element in A and show that it is in B
2) show that $B \subseteq A$, i.e. choose an arbitrary element in B and show that it is in A

The element was chosen arbitrary, hence any element that is a member of the left set, is also a member of the right set, and vice versa.

Example:
Prove that $A - B = A \cap \sim B$

1. Show that $A - B \subseteq A \cap \sim B$

   Let $x \in A - B$
   By Def 3:
   $x \in A \land x \not\in B$ (1)
   By (1) $x \in A$ (2)
   By (1) $x \not\in B$ (3)
   By (3) and Def 4: $x \in \sim B$ (4)
   By (2), (4)
   $x \in A \land x \in \sim B$ (5)
   By (5) and Def 2:
   $x \in A \cap \sim B$

x was an arbitrary element in $A - B$, therefore $A - B \subseteq A \cap \sim B$ (6)
2. Show that \( A \cap \neg B \subseteq A - B \)

Let \( x \in A \cap \neg B \)

By Def 2:
\[
x \in A \land x \in \neg B
\]

By (7) \( x \in A \)

By (7) \( x \in \neg B \)

By (9) and Def 4: \( x \notin B \)

By (8), (10)
\[
x \in A \land x \notin B
\]

By (11) and Def 3:
\[
x \in A - B
\]

\( x \) was an arbitrary element in \( A \cap \neg B \), therefore \( A \cap \neg B \subseteq A - B \) (12)

by (6) and (12):
\[
A - B \subseteq A \cap \neg B
\]

Q.E.D.

**B. Using set identities**

Prove that \( A \cap (\neg A \cup B) = A \cap B \)

**Method:** Apply the set identities to the expression on the left, until the expression on the right is obtained.

By Distribution Laws:
\[
A \cap (\neg A \cup B) = (A \cap \neg A) \cup (A \cap B)
\]

By the Exclusion Law
\[
A \cap \neg A = \emptyset
\]

Hence
\[
A \cap (\neg A \cup B) = \emptyset \cup (A \cap B)
\]

By the Identity Law:
\[
\emptyset \cup (A \cap B) = A \cap B
\]

Hence
\[
A \cap (\neg A \cup B) = A \cap B
\]

Q.E.D.
The story of Q.E.D.

"quod erat demonstrandum" (literally, "which was to be demonstrated"). Latin phrase, written at the end of Math proofs.

In French this acronym translates by C.Q.F.D (literally "ce qu'il fallait démontrer").

In Spanish it translates by L.Q.Q.D., (i.e. "lo que queda demostrado").

In Italian it translates by C.V.D. ("come volevasi dimostrare").

End-of-proof symbol in the present day is often the symbol ■ (solid black square) called **tombstone** or Halmos symbol. used by Paul Richard Halmos (1916 – 2006) American mathematician (born in Hungary)

In English speaking countries the letters have been humorously interpreted as

"Quite Easily Done" or
"Quickly and Easily Done" or, occasionally,
"Quite Eloquently Done", or
"Quite Enough Done".

Other humorous expansions in the context of mathematical proofs are

"Question Every Detail" or
"Question Every Deduction", suggesting that the reader should check that the proof is indeed correct as claimed, or
"Qualitatively Extracted Deduction" or
"Quit and Eat Dinner."

It has also been rendered as *quod ego dico*, "because I say so" or
*quo errat demonstrator*, "where the demonstrator erred".

Alternatively, some people prefer to use **WWWWW** or **W**° which stands for the English
"Which Was What Was Wanted." or
"Which Was What We Wanted."