Finite and Infinite Sets. Countability. Proof Techniques
(Chapter 1, Sections 1.4, 1.5)

**Finite sets** - finite number of elements - can be counted

**Infinite sets** - infinite number of elements

**Equinumerous sets:**

Sets A and B are equinumerous if there is a bijection $f : A \rightarrow B$

E.G.: $A = \{1, 2, 3\}$, $B = \{a, b, c\}$

**Countably infinite sets:**

A set is countably infinite if it is equinumerous with N (the set of natural numbers)

**Countable sets**

Finite sets and countably infinite sets are countable.
Sets that are not countable, are called uncountable.

**Proving countability/uncountability of sets:**

Three basic techniques:

a. finding a **bijection** to the set of natural numbers
b. applying the **dovetailing** technique
c. using the **diagonalization** principle

These techniques may be used in a combined way, together with direct proof and proof by contradiction

**A. Bijection**

**Example 1:** The set of all squares of natural numbers is countably infinite

$A = \{k^2 : k \in \mathbb{N}\}$, i.e. $A = \{1, 2, 4, 9, \ldots\}$

We can construct a bijection $f : A \rightarrow B$

$f(k^2) = k$

Hence A is equinumerous to N.
B. Dovetailing method

Example 2:

The union of a finite number of countably infinite sets is countably infinite

Let \( A = \{a_1,a_2,a_3, \ldots\} \)
\( B = \{b_1,b_2,b_3, \ldots\} \),
\( C = \{c_1,c_2,c_3\ldots\} \)
(we can order the elements of A, B and C, as each of them is countably infinite)

Let A, B, and C be pairwise disjoint sets (the proof in the general case is similar)
Consider the set \( A \cup B \cup C \)

To prove that is it countably infinite, we need to find a way to order the elements.
If we start counting \( a_1, a_2, a_3, \ldots \) we will never come to the elements of B and C.
We need something else, and it is the following:

\( a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, \ldots \)

We count first all elements with index 1, then we count all elements with index 2, etc.

**Important:** it is possible to count all the first elements (and then the second, third, etc) because we have only 3 sets: A, B, C, i.e. finite number of sets, so we can put the sets themselves (not the elements of the sets) in order. Here we have chosen A, B, C. It is possible to choose also B, A, C, C, B, A, etc.

**Generalization:**

If we have \( m \) sets \( A_1, A_2, A_m \), that are countably infinite, their union \( A_1 \cup A_2 \cup \ldots \cup A_m \) will also be countably infinite, for any \( m > 1 \).

This can be proved using mathematical induction.
1. **Inductive base:**
   - The union of two countably infinite sets is countable.
   - We order the elements in this way: \( a_1, b_1, a_2, b_2, a_3, b_3, \ldots \).

2. **Assume that the union of \( k \) number of countably infinite sets is countable.**
3. **We can show that the union of \( k+1 \) number of countably infinite sets is countable.**

   Let \( T = A_1 \cup A_2 \cup \ldots \cup A_k \)

   Following our assumption we can order the elements in \( T \):
   \( t_1, t_2, t_3, t_4, \ldots \)
Let \( A = \{a_1, a_2, a_3, \ldots\} \) is countably infinite
Consider the union \( T \cup A \).

We can count the element is the following way:
\( t_1, a_1, t_2, a_2, t_3, a_4, \ldots \)

What if we had \textit{infinite number} of countably infinite sets?
As we'll see, in some cases it is infinitely countable, in other cases it is uncountable.

**Example 3:**

The Cartesian product \( N \times N \) is a countably infinite set.

Consider the elements of \( N \times N \) put in a table in the following way:

\[
\begin{array}{cccccccc}
  (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) & \ldots \ldots \\
  (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) & \ldots \ldots \\
  (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) & \ldots \ldots \\
  (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) & \ldots \ldots \\
  (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) & \ldots \ldots \\
  \ldots \ldots \\
\end{array}
\]

So, the pair \((a,b)\) in \( N \times N \) occupies the \( a^{th} \) row and the \( b^{th} \) column.

**Step 1:** First we visit the first element in the first row \((1,1)\)
**Step 2:** Then we visit the first element in the second row \((2,1)\) and the second element in the first row \((1,2)\)
**Step 3:** Then we visit the first element in the third row \((3,1)\), the second element in the second row \((2,2)\) and then the third element in the first row \((3,1)\)

\[\ldots \ldots\]

**Step k:** We visit the first element in the \( k^{th} \) row \((k,1)\), then the second element in the \((k-1)\) row \((k-1,2)\), then the third element in the \((k-2)\) row: \((k-2,3)\), and so on until we reach the \( k^{th} \) element in the first row \((1,k)\)

**Note** that we can visit the elements in somewhat opposite order: **Step k** would be:
Visit the \( k^{th} \) element in the first row \((1,k)\), then the \((k-1)^{st}\) element in the second row \((2,k-1)\), etc, until we reach the first element in the \( k^{th} \) row \((1,k)\)

Thus we show that \( N \times N \) is countably infinite.
The essence: What matters here is that N is countably infinite (by its definition - this is the set of all natural numbers) so we can order its element and build a table with rows and columns corresponding to that order.

If A = \{a_1, a_2, a_3, \ldots\} is a countably infinite set, and B = \{b_1, b_2, b_3, \ldots\} is a countably infinite set, then
The set A \times B is countably infinite.

We can construct a table where the first row corresponds to a_1, the second to a_2, etc, the first column corresponds to b_1, the second - to b_2, etc.
Then each element in the table would correspond to an element in A \times B, and we can use the above technique of visiting the elements of A \times B in order.

Example 4:

The Cartesian product N \times N \times N is a countably infinite set.

N \times N \times N = (N \times N) \times N.
We already know that N \times N is countably infinite, so we can order its elements.
The elements of N \times N \times N can be represented in a table, where each column corresponds to a natural number 1, 2, 3, etc, and each row corresponds to the an element in N \times N:

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

With this table we can proceed in the same way as for N \times N:
The first elements in the ordering would be:

(1,1,1) (2,1,1) (1,1,2) (1,2,1) (2,1,2) (1,1,3) ….

We can generalize and say that the Cartesian product of a finite number of countably infinite sets is countably infinite. (using mathematical induction)
What about if we had a Cartesian product of infinite number if countably infinite sets? Using the diagonalization principle, we can show that it is not countable.

**Example 5:**

Let's come back to the union of infinite number of countably infinite sets. **The issue here is:** can we put the elements of the union in a table? If we can - then the union is countable, otherwise it is not.

We are looking at the union of $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k \cup \ldots$.

Let's see what we have:

- $A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}$
- $A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\}$
- $A_3 = \{a_{31}, a_{32}, a_{33}, a_{34}, \ldots\}$
- $\ldots$
- $A_k = \{a_{k1}, a_{k2}, a_{k3}, a_{k4}, \ldots\}$
- $\ldots$

At first sight it seems that we can put the elements in a table, just using their indices:

\[
\begin{array}{cccc}
  & a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  \vdots & \vdots & \vdots \\
\end{array}
\]

Now, I have named the sets $A_1, A_2, \ldots$ which means that I have implicitly ordered them in some way. However is this always possible? Let us consider the set $R = \{A_1, A_2, \ldots, A_k, \ldots\}$ This is an infinite set. It may be countable, or it may be not countable.

Only if the set is countable, we can order the sets $A_1, A_2, A_3, \ldots$ and then use the dovetail technique to order the elements in the union. Otherwise we would not be able to order the sets and assign a row to each set. Then the union is not countable.

Note: The set $S_1 = A_1 \cup A_2 \cup A_3 \cup \ldots$ is different from the set $S_2 = \{A_1, A_2, A_3, \ldots\}$. $S_1$ is countable iff $S_2$ is countable and each $A_i$ is countable.
C. Diagonalization method

Let $A$ be a finite set, and $R$ be a binary relation on $A$. We can represent the relation by a square table, rows and columns representing the elements, and cells having 1s if there is a link between the corresponding elements.

For example, if $A = \{a,b,c,d\}$, and $R = \{(a,b), (a,c), (b,b), (b,d), (c,b), (c,d), (d,a)\}$, the table would be:

\[
\begin{array}{cccc}
  & a & b & c & d \\
 a & 0 & 1 & 1 & 0 \\
 b & 0 & 1 & 0 & 1 \\
 c & 0 & 1 & 0 & 1 \\
 d & 1 & 0 & 0 & 0 \\
\end{array}
\]

The principle says that the compliment of the diagonal (replacing 1s with 0s and vice versa) is different from each row.

The reversed diagonal in the example is 1,0,1,1 and you can see that it is different from each row.

**Explanation:** The reversed diagonal differs from the first row in the first element (we have taken the complement in the diagonal), it differs from the second row in the second element, etc.

**Theorem 1.5.2.** The set $2^\mathbb{N}$ is uncountable

**Proof:**

Let $2^\mathbb{N} = \{R_1, R_2, \ldots \}$ is the power set of $\mathbb{N}$, it contains all possible subsets of $\mathbb{N}$. We can represent each subset as a sequence of 0s and 1s, where the $i^{th}$ position is 1 if $i$ is in the subset, and 0 otherwise.

Let us assume that $2^\mathbb{N}$ is countable. This means that there is a bijection between $2^\mathbb{N}$ and $\mathbb{N}$, so we can order the elements of $2^\mathbb{N}$ accordingly. In the ordered sequence we represent each $R_k$ by means of 0s and 1s. Thus we obtain an infinite table filled with 0s and 1s.

Consider now the reverse diagonal $D = \{d_1, d_2, \ldots \}$ of that table. It is a sequence of 0s and 1s, so it corresponds to some set of natural numbers $D = \{n : d_n = 1\}$

This set obviously is a subset of $\mathbb{N}$, so it should be in the power set of $\mathbb{N}$, and should appear somewhere in the ordered elements of $2^\mathbb{N}$, i.e. should be equal to some row in our table.
However, D cannot be the first row as it differs from the first row in the first position. It cannot be also equal to the second row as it differs there in the second position, etc.

So D cannot be anywhere among the rows of the table. However we have assumed that the table contains all possible subsets of N. This is a contradiction, following from our assumption, that the elements of $2^N$ can be ordered. Hence $2^N$ is uncountable.

D. Direct Proof and Proof by Contradiction

Example 6:

If A is countably infinite, and B is a subset of A, then B is countable too.

A is countable, so we can order its elements. The elements of B are also in A, so they all appear somewhere in the ordering, i.e. they are also ordered. Hence B is countable.

Example 7:

If A is uncountable its union with any other set is uncountable too.

Let B be a set, and $C = A \cup B$.

Let us assume that C is countable. A is a subset of C, and we have proved, that a subset of countable set must be countable. Hence A must be countable. Thus we have arrived to a contradiction, that follows from our assumption. Hence C is uncountable.

Example 8:

If A is uncountable set and B is countable, the set $C = A - B$ is uncountable.

Proof:

$A = (A - B) \cup (A \cap B)$  (Draw the Venn diagram to see this)

$(A \cap B) \subseteq B$, hence it is countable.

We have shown that the union of countable sets is countable. If we assume that A-B is countable, this will mean that A is countable. Thus we come to a contradiction. Hence A-B is uncountable.
What you have to know:

1. Infinite sets are countable or uncountable.
2. To show that an infinite set is countable, we have to do one of the following:
   a. to find a bijection between the set and N (natural numbers)
   b. to show that the elements can be placed in a table and to use the dovetailing counting.
   c. To show that a set is uncountable we use the diagonalization method.
3. Any subset of a countable set is countable
4. The union of finite number of countable sets is countable
5. The Cartesian product of finite number of countable sets is countable
6. The intersection of finite number of countable sets is countable.
7. If A is uncountable and B is countable, then
   A - B is uncountable
   A ∩ B is countable
   A ∪ B is uncountable
   A x B is uncountable

Some insights:

1. The pigeonhole principle: there is no one-to-one function from a larger set to a smaller one.
2. The diagonalization principle: applied to infinite number of infinite sets
   Works to prove that $2^N$ is uncountable - this is the power set, containing all subsets, both finite and infinite.
   Does not work for the set of all finite subsets
3. Compare the dovetailing method of counting and the diagonalization principle.

   Both deal with tables.

   The dovetailing technique is applied when we have in advance the ordering of the rows and the columns. Then we can show how to count the elements in the table.

   The diagonalization principle is used to show that we cannot order the rows in cases when the reverse diagonal needs to be one of the rows. If there is no need for the reversed diagonal to be among the rows, there is no contradiction and we cannot use this principle to show that the rows cannot be ordered.